[This question paper contains 4 printed pages.]

(10)

Your Roll No. 2022

Sr. No. of Question Paper : 1132 A

Unique Paper Code : 32351201

Name of the Paper : BMATH203 – Real Analysis

Name of the Course : **B.Sc. (H) Mathematics**

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.
4. **All** questions carry equal marks.

1. (a) Let S be a non- empty bounded set of $\mathbb{R}$. Let $b<0$ and let $bS=\{bs \mid s\in S\}$. Prove that $inf(bS) = b\,sup\,S$ and $sup(bS) = b\,inf\,S$.

   (b) If y is a positive real number, show that there exists $n_y \in \mathbb{N}$ such that

   $$n_y - 1 \le y < n_y$$

   (c) Let X be a non- empty set. Let f and g be defined on $\mathbb{R}$ and have bounded ranges in $\mathbb{R}$. Show that

   $$\sup\{f(x)+g(x) \mid x\in X\} \le \sup\{f(x) \mid x\in X\} + \sup\{g(x) \mid x\in X\}.$$

P.T.O.

(d) Define a sequence $\langle e_n \rangle$ by $e_n = \left(1+\frac{1}{n}\right)^n$, $\forall n \in \mathbb{N}$.

Show that $\langle e_n \rangle$ is bounded and increasing and hence converges. Also, show that $\lim \langle e_n \rangle$ Lies between 2 and 3.

2. (a) State and prove Density theorem.

(b) Let A and B be bounded non- empty subsets of $\mathbb{R}$ and let $A+B=\{a+b \mid a \in A, b \in B\}$. Prove that $\sup(A+B)=\sup A+\sup B$ and $\inf(A+B)=\inf A+\inf B$

(c) Let $I_n = \left[0, \frac{1}{n}\right]$, $n \in \mathbb{N}$. Show that $\{I_n, n \in \mathbb{N}\}$ is a nested sequence of intervals and $\bigcap_{n \in \mathbb{N}} I_n = \{0\}$.

(d) Examine the convergence of the series $\sum_{n=1}^{\infty} ne^{-n^2}$

3. (a) State and prove Monotone Convergence Theorem.

(b) Let (x_n) be a sequence of positive real numbers such that $\lim_{n\to\infty} (x_n^{1/n}) = \mathrm{L}$ exists. Prove that if $\mathrm{L} < 1$, then (x_n) converges and $\lim_{n\to\infty} (x_n) = 0$.

(c) Prove that $\lim_{n\to\infty} (n^{\frac{1}{n}}) = 1$.

(d) Use the definition of the limit to show that $\lim_{n\to\infty} (x_n) = 0$, where $x_n = 1/\ln(n+1)$, for $n \in \mathbb{N}$. Also find $K \in \mathbb{N}$ for $\varepsilon = \frac{1}{10}$ such that $|x_n - 0| < \varepsilon, \forall n \geq K$.

4. (a) Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that converge to x and y respectively and if $y \neq 0$. Then the quotient sequence X/Y converges to x/y.

(b) State and prove Squeeze Theorem. Also find $\lim_{n\to\infty} (\frac{\sin n}{n})$

(c) State Cauchy Convergence Criterion for Sequences.
Let $X = (x_n)$ be defined by $x_1 = 1, x_2 = 2$ and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for $n > 2$. Prove that the sequence X is convergent.

(d) Discuss the convergence of the sequence (x_n), where
$$x_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots\cdots + \frac{1}{n^2}, \text{ for each } n \in N.$$

5. (a) Suppose the k th partial sum of $\sum_{n=1}^{\infty} x_n$ is $s_k = \frac{k}{k+1}$. Find the corresponding series and general term x_n. Prove that the series converges and then find the limit.

(b) Prove that the harmonic series $\sum \frac{1}{n}$ diverges (despite the fact that $\lim \frac{1}{n} = 0$).

(c) Test for convergence, the following series:

(i) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ (ii) $\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11} + \cdots$

(d) Show that the series $1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, p > 0$ converges absolutely for $p > 1$ and conditionally for $0 < p \leq 1$.

6. (a) Prove that if $\sum_{n \geq 1} a_n$ is a series of positive terms and that its partial sums are bounded, then $\sum_{n \geq 1} a_n$ converges. Show that this is not necessarily true if $\sum_{n \geq 1} a_n$ is not a series of positive terms.

(b) State and prove the limit comparison test.

(c) Test for convergence, the following series:

(i) $1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^n} + \cdots$

(ii) $\sum_{n=1}^{\infty} 3^{-n-(-1)^n}$

(d) Define absolute and conditional convergence of an alternating series. Show that the series $\sum \frac{(-1)^{n+1}}{\sqrt{n}}$ is conditionally convergent but not absolutely.

[This question paper contains 6 printed pages.]

(11) Your Roll No. 2022

Sr. No. of Question Paper : 738 **B**

Unique Paper Code : 32351201

Name of the Paper : BMATH203 – Real Analysis

Name of the Course : **B.Sc. (H) Mathematics**

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All Questions are Compulsory.
3. Attempt any **two** parts from each question.
4. All Questions are of equal marks.

1. (a) State the completeness property of $\mathbb{R}$, hence show that every non-empty set of real numbers which is bounded below, has an infimum in $\mathbb{R}$.

(b) Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set and $\sup (A \cup B) = \max \{\sup A, \sup B\}$.

(c) State and prove nested interval property.

(d) Define an open set and closed set in $\mathbb{R}$.

Show that if $a, b \in \mathbb{R}$, then the open interval (a, b) is an open set.

Is a closed interval a closed set ?

2. (a) Let S be a bounded set in $\mathbb{R}$ and let S_0 be a non-empty subset of S. Show that

$$\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S$$

(b) State Archimedean property. Hence, prove that if

$S = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$ then $\inf S = 0$.

(c) If $S \subseteq \mathbb{R}$ is non empty. Show that S is bounded if and only if there exists a Closed bounded interval I such that $S \subseteq I$.

(d) If $x, y, z \in \mathbb{R}$ and $x \leq z$. Show that $x \leq y \leq z$ if and only if $|x-y| + |y-z| = |x-z|$. Interpret this geometrically.

3. (a) Prove that a convergent sequence of real numbers is bounded.

Is the converse true? Justify.

(b) Let (x_n) be a sequence of positive real numbers such that $\lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right) = L$ exists. If $L < 1$, then (x_n) converges and $\lim_{n\to\infty}(x_n) = 0$.

(c) Prove that if $C > 0$, then $\lim_{n\to\infty}\left(C^{1/n}\right) = 1$.

(d) Let $x_1 > 1$ and $x_{n+1} = 2 - \frac{1}{x_n}$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Also find the limit.

4. (a) Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that converge to x and y respectively. Then the product sequence X.Y converges to x.y.

(b) Let $X = (x_n)$ be a bounded sequence of real numbers and let $x \in R$ have the property that every convergent subsequence of X converges to x. Then the sequence X is convergent to x.

(c) Discuss the convergence of the sequence (x_n),

where $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots\cdots + \frac{1}{2n}$ for $n \in N$.

(d) Use the definition of the limit of the sequence to find the following limits

(i) $\lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

(ii) $\lim_{n\to\infty} \left(\frac{3n+1}{2n+5}\right)$

5. (a) Prove that a necessary condition for the convergence of an infinite series Σa_n is $\lim_{n\to\infty} a_n = 0$. Is the condition sufficient? Justify with the help of an example.

(b) Prove that the geometric series $1 + r + r^2 + \cdots$ converges for $0 \le r \le 1$ and diverges for $r \ge 1$.

(c) Test for convergence, the following series :

(i) $\frac{1}{5}+\frac{2!}{5^2}+\frac{3!}{5^3}+\cdots$

(ii) $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

(d) Prove that the series $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$ converges if and only if $-1 \le x \le 1$.

6. (a) State and prove Cauchy's n^{th} root test for positive term series.

(b) Prove that the series $1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}\cdots$ converges for $p > 1$ and diverges for $p \le 1$.

(c) Test for convergence, the following series :

(i) $\sum_{n=1}^{\infty}\left[\sqrt[3]{n^3+1}-n\right]$

(ii) $\sum_{n=1}^{\infty} 2^{-n-(-1)^n}$

(d) Prove that every absolutely convergent series is convergent. Show that the series $\sum (-1)^n \frac{n+2}{2^n+5} x^n$ converges for all the real values of x.

[This question paper contains 8 printed pages.]

(12) Your Roll No. 2022

Sr. No. of Question Paper : 756 **B**

Unique Paper Code : 32351202

Name of the Paper : Differential Equation

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Use of non-programmable scientific calculator is allowed.

1. Attempt any **three** parts. Each part is of 5 marks.

(a) Solve the differential equation

$$(x^3 + y^3)dx - (x^2y + xy^2)dy = 0.$$

(b) Solve the initial value problem

$$(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0, \ y(0) = 1.$$

(c) Find the general solution of the differential equation

$$xy'' + y' = 4x.$$

(d) Solve the differential equation

$$(x^3y^2 + xy)dx = dy.$$

(e) Find an integrating factor of the form $x^p y^q$ and solve the differential equation

$$(8x^2y^3 - 2y^4)dx + (5x^3y^2 - 8xy^3)dy = 0.$$

2. Attempt any **two** parts. Each part is of **6** marks.

(a) Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei has undergone disintegration in a period of 100 years.

(i) What percent of the original radioactive nuclei will remain after 1000 years?

(ii) In how many years will only one-fourth of the original number remain?

(b) A certain city had a population of 25,000 in 1960 and a population of 30,000 in 1970. Assume that its population will continue to grow exponentially at a constant rate. What population can its city planner expect in the year 2000?

(c) An arrow is shot straight upward from the ground with an initial velocity of 160 ft/s. It experiences both the deceleration of gravity and deceleration $v^2/800$ due to air resistance. How high in the air does it go?

(d) The following differential equation describes the level of pollution in the lake

$$\frac{dC}{dt} = \frac{F}{V}C_{in} - C$$

where V is the volume F is the flow (in and out), C is the concentration of pollution at time t and C_{in} is the concentration of pollution entering the lake. Let $= 28 \times 10^6$ m^3, $F = 4 \times 10^6$ m^3/month. If only freshwater enters the lake.

(i) How long would it take for the lake with pollution concentration 10^7 parts/m^3 to draw below the safety threshold 4×10^6 parts/m^3?

(ii) How long will it take to reduce the pollution level to 5% of its current level?

3. Attempt any **two** parts. Each part is of 6 marks.

(a) Write down the word equations along with compartment diagrams that describe the movement of the drugs between the three compartments in the body, the GI tract, the bloodstream and the urinary tract, when a patient takes a single pill. Here, the urinary tract is only an absorbing compartment. From the word equations, develop the differential equation system which describes this process, defining all variables and parameters as required.

(b) Solve the logistic differential equation with the initial condition $X(0) = x_0$.

(c) A population, initially consisting of 1000 mice, has a per capita birth rate of 8 mice per month (per mouse) and a per capita death rate of 2 mice per month (per mouse). Also, 20 mouse traps are set each week and they are always filled.

(i) Write down a word equation describing the rate of change in the number of mice and hence write down a differential equation for the population.

(ii) Find the population of mice after 6 months.

(d) Consider the harvesting model

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - h.$$

(i) Find the two non-zero equilibrium populations.

(ii) If the harvesting rate h is greater than some critical value h_c, the non-zero equilibrium values do not exist and the population tends to extinction. What is this critical value h_c?

(iii) If the harvesting rate $h < h_c$, the population may still extinct if the initial population x_0 is below some critical level X_c. What is this critical initial value X_c?

4. Attempt any **two** parts. Each part is of **6** marks.

(a) Find the general solution of the differential equation

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} - 10x\frac{dy}{dx} - 8y = 0.$$

(b) Show that $y = 1/x$ is a solution of $y' + y^2 = 0$, but that if $c \neq 0$ and $c \neq 1$, then $y = c/x$ is not a solution.

(c) Use the Wronskian to prove that the functions

$$f(x) = e^x,\ g(x) = e^{2x},\ k(x) = e^{3x};$$

are linearly independent on the real line.

(d) Use method of undetermined coefficient to find particular solution of differential

$$y''' + y' = 2 - \text{Sin } x.$$

5. Attempt any two parts. Each part is of 6 marks.

(a) Find the general solution of the differential equation

$$y^{(4)} - 8y'' + 16y = 0$$

(b) Solve the initial value problem

$$2y^{(3)} - 3y'' - 2y' = 0;\ y(0) = 1,\ y'(0) = -1,\ y''(0) = 3.$$

(c) Find the general solution of the Euler's equation

$$x^3y''' - 3x^2y'' + xy' = 0.$$

(d) Use the method of variation of parameters to find the solution of the differential equation

$$y'' + 3y' + 2y = 4e^x.$$

6. Attempt any two parts. Each part is of 6 marks.

(a) By making appropriate assumptions develop a model with two differential equations describing predator-prey interaction with DDT spray effect. Check the model in the limiting case of

(i) Prey with no predator.

(ii) Predator with no prey.

(b) In a long-range battle, neither army can see the other, out fires into a given area. A simple mathematical model describing this battle is given by the coupled differential equations

$\frac{dR}{dt} = -c_1 RB$, $\frac{dB}{dt} = -c_2 RB$, where R: Red Army, B: Blue Army where c_1 and c_2 are positive constants.

(i) Use the chain rule to find a relationship between R and B, given the initial numbers of soldiers for the two armies are r_0 and b_0, respectively.

(ii) Draw a sketch of typical phase-plane trajectories.

(c) Suppose that soldiers from the red army are visible to the blue army, but soldiers from the blue army are hidden. Thus, the red army is using random firing while the blue army uses aimed firing

(i) Write down appropriate word equations describing the rate of change of the number of soldiers in each of the armies.

(ii) By making appropriate assumptions, obtain two coupled differential equations describing this system.

(iii) Extend the model to include reinforcements if both of the armies receive reinforcements at a constant rate.

(d) The pair of differential equations

$$\frac{dP}{dt} = rP - \gamma PT, \qquad \frac{dT}{dt} = qP,$$

where r, γ and q are positive constants, is a model for a population of microorganisms P, which produces toxins T which kill the microorganisms.

(i) Given that initially there are no toxins and p_0 microorganisms, obtain an expression relating the population density and the amount of toxins.

(ii) Give a sketch of a typical phase-plane trajectory, indicating the direction of movement along the trajectories.

(iii) Using this model, describe what happens to the microorganisms over time.